

NON-LINEAR DYNAMICS OF NON-NEUTRAL PLASMAS

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Abstract—An expanding or contracting non-neutral plasma without an external magnetic field is investigated. In certain situations the density and temperature rise to many times their initial values in a short time, followed by an equally sharp fall. The plasma shows no Landau damping.

1. INTRODUCTION

NON-NEUTRAL PLASMAS have been investigated since 1970. The trend-setting papers (DAVIDSON and KRALL, 1970; DAVIDSON, 1971, 1974; DOUGLAS and O'NIEL, 1978; O'NIEL, 1980, 1981; DRISCOLL *et al.*, 1983) have been followed by many key papers (CRAWFORD *et al.*, 1985; CRAWFORD and O'NIEL, 1986; O'NIEL and HJORTH, 1985; O'NIEL, 1988; DAVIDSON, 1988; DUBIN and O'NIEL, 1988). In the literature the effect of expansion and contraction on non-neutral plasma has not attracted much attention in the past. Therefore, in this paper we report the results of an investigation on single- and two-species non-neutral plasmas. Each charged particle species has a drift velocity which is a function of space and time. The density and the temperature, however, are taken to be functions of time only.

2. SINGLE-SPECIES PLASMA

In the absence of an external magnetic field, let the particle distribution be given by

$$f(\mathbf{r}, \mathbf{v}, t) = \frac{C(t)}{\{2\pi B_1(t)B_2(t)B_3(t)\}^{3/2}} \exp \left[- \left[\frac{[v_x - u_x]^2}{2B_1(t)} + \frac{[v_y - u_y]^2}{2B_2(t)} + \frac{[v_z - u_z]^2}{2B_3(t)} \right] \right], \quad (1)$$

where $C(t)$ and B are the time-dependent density and square of the thermal velocities (in three directions). We choose the spatial dependence of \mathbf{u} to be of the form

$$\mathbf{u}(\mathbf{r}, t) = [xA_1(t), yA_2(t), zA_3(t)], \quad (2)$$

where A_i are the unknown functions of time.

Substituting (1) into Vlasov's equation, using Poisson's equation for the electrostatic field and equating the various powers of v_i to zero yields, after some manipulations,

$$B_j^{-1} \frac{dB_j}{dt} + 2A_j = 0, \quad j = 1, 2, 3; \quad (3)$$

$$\frac{dA_j}{dt} + A_j^2 - 4\pi \frac{e^2}{3m} = 0; \quad (4)$$

$$C^{-1} \frac{dC}{dt} + (A_1 + A_2 + A_3) = 0; \quad (5)$$

where e and m are the charge and the mass of the species. We shall assume the initial values of the A_j to be the same, in which case \mathbf{A} becomes isotropic because of (4). Since \mathbf{A} is isotropic, equation (3) implies B is isotropic. Therefore, let $A_1 = A_2 = A_3 = A$, $B_1 = B_2 = B_3 = B$. One should note that \mathbf{u} and the electrostatic field are radial and that no self-magnetic field can be generated. The whole plasma contracts when A is < 0 and expands when $A > 0$. Equations (3)–(5) can also be obtained by taking the first three moments of Vlasov's equation. Actually, (5) is the continuity equation conserving particles, (4) is the equation of momentum transfer and (5) is the energy equation. The question naturally arises as to why one did not take the moments of Vlasov's equation to obtain the above equations directly. There is a simple answer to this. Any other distribution having the same mean characteristics as distribution (1) will also lead to equations (3)–(5), but such a distribution may not satisfy Vlasov's equation, i.e. the solution may be true fluid dynamically but not kinetically. It is, therefore, obvious that the present solution is kinetically correct. Now we shall define the following dimensionless variables to solve the above equations:

$$\tau = t\omega_0, \quad \tilde{A} = \frac{A}{\omega_0}, \quad \tilde{B} = \frac{B}{B(0)}, \quad (6)$$

where ω_0 is the plasma frequency of the species corresponding to the density $C(0)$. Using these definitions, equations (4) and (5) reduce to

$$\frac{d\sigma}{d\tau} + 3\sigma\tilde{A} = 0, \quad (7)$$

$$\frac{d\tilde{A}}{d\tau} + \tilde{A}^2 - \frac{\sigma}{3} = 0. \quad (8)$$

From (7) to (8), we obtain a single equation for $\sigma(\tau)$:

$$\frac{d^2\sigma}{d\tau^2} - \frac{4}{3} \left[\frac{d\sigma}{d\tau} \right]^2 + \sigma^2 = 0. \quad (9)$$

The first integration yields

$$\frac{d\sigma}{d\tau} = [K_0\sigma^{8/3} - 6\sigma^3]^{1/2}, \quad (10)$$

where K_0 is a constant of integration. Substituting

$$\sigma^{1/3} = \frac{K_0}{6} \sin^2 \theta \tag{11}$$

and integrating (10) yields

$$\left\{ \ln \tan \frac{\theta}{2} - \frac{\cos \theta}{\sin^2 \theta} \right\} - \left\{ \ln \tan \frac{\beta}{2} - \frac{\cos \beta}{\sin^2 \beta} \right\} = (18)^{-1} K_0^{3/2} \tau, \tag{12}$$

where $\beta = \theta(0)$. The above equation thus give the variation of θ as function of τ . Looking at it in more detail, it can be seen that when $d\sigma/d\tau > 0$ at $\tau = 0$, β lies in the first quadrant. Hence $\sigma(\tau)$ first increases until θ reaches $\pi/2$, and thereafter decreases until θ reaches π , which corresponds to $\tau \rightarrow \infty$. Thus the plasma contracts for a certain duration of time and then continues to expand forever. This phenomenon can be qualitatively understood as follows. From equations (3) and (7), the radial velocity equals $\omega_0/3 (d/d\tau \ln \sigma)\mathbf{r}$. Therefore initially when the time derivative of $\sigma(\tau)$ is positive, the radial motion is inwards against the electrostatic force. The inward motion continues until the electrostatic force is able to overtake this motion and once that happens there is no way to reverse it because the force due to the electric field is always radially outwards. Hence the plasma expands forever.

Let $(d\sigma(0)/d\tau) = \alpha$, and $\tau_c =$ the time required to reach the maximum density. From (12), τ_c can be calculated by putting $\theta = \pi/2$ and is given by

$$\tau_c = \frac{\cos \beta}{\sin^2 \beta} - \ln \left(\tan \frac{\beta}{2} \right) \frac{18}{(\alpha^2 + 6)^{3/2}}. \tag{13}$$

A plot of τ_c versus α is given in Fig. 1. It can be seen that for large α , τ_c decreases. For example, for $\alpha = 3$ the plasma is moving radially inwards with velocity $\omega_0 r$. Then $\sigma_{\max} \approx 18\sigma(0)$ is obtained in a time $\tau_c \approx \omega_0^{-1}$, whereas for $\alpha = 16$, $\sigma_{\max} \approx 5 \times 10^4 \sigma(0)$ is obtained in about $0.21 \times \omega_0^{-1}$. If, on the other hand, $\alpha < 0$, then β lies in the second quadrant and the density decreases until $\theta \rightarrow \pi$ ($\tau \rightarrow \infty$). Even though equation (12) helps us to understand the time evolution of the density qualitatively, it is very difficult to solve (12) for $\sigma(\tau)$ and $A(\tau)$. Hence (7) and (8) have been integrated numerically

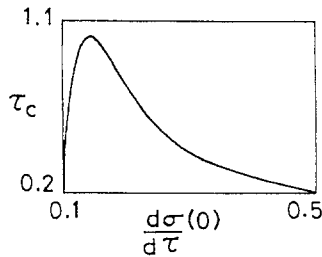


FIG. 1.—Dimensionless time τ_c taken to reach maximum density versus initial values of $d\sigma/d\tau$ for a single-species plasma.

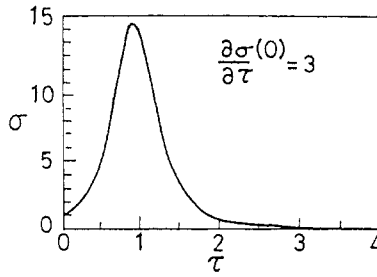


FIG. 2.—Time τ versus density σ for $d\sigma(0)/d\tau = 3$.

and the results for $\alpha = 3$ and 5 are shown in Figs 2–5. For other values of α the solutions remain qualitatively unchanged. The case of non-radial velocity field also shows similar behavior.

Let us calculate the minimum radius of the species mass in terms of its radius at $\tau = 0$ as the density becomes maximum. The equation of motion gives

$$\frac{dr}{d\tau} = Ar = -\frac{1}{3\sigma} \frac{d\sigma}{d\tau} r, \tag{14}$$

where (7) has been used. Integrating this equation, one obtains

$$r = \frac{r_0}{\sigma^{1/3}}, \tag{15}$$

where r_0 is the radius at $t = 0$. Putting $\theta = \pi/2$ in (11) one gets the maximum density, which together with (15) yields the minimum radius as

$$r_{\min} = 6r_0(6 + \alpha^2)^{-1}. \tag{16}$$

For previously considered values of $\alpha = 3, 5$, r_{\min} turns out to be in the neighbourhood of $0.4r_0$ and $2.6 \times 10^{-2}r_0$, respectively. Knowing $\sigma(\tau)$, it is quite easy to obtain $B(\tau)$,

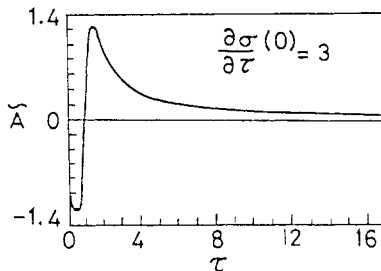


FIG. 3.— τ versus \tilde{A} for $d\sigma/d\tau = 3$.

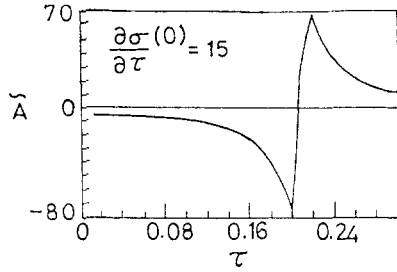


FIG. 4.— τ versus σ for $d\sigma/d\tau(0) = 5$.

because from (3) and (5) one obtains

$$\frac{B(\tau)}{B(0)} = [\sigma(\tau)]^{2/3}. \tag{17}$$

Therefore, an increase in the density results in an increase in the temperature. From (15) and (17) one can also see that the adiabatic index of expansion is $5/3$, as expected.

3. ELECTRON-ION PLASMA

We shall neglect the dynamics of the positively charged ions which are assumed to be heavy. The electron distribution, together with the drift velocity, will continue to be given by eqns (1) and (2) as before, but the electron density and the square of the thermal velocity will be replaced by C_e and Θ_i respectively. In the definition for the electron drift velocity we shall replace A_i by a_i . As before, the electrostatic field will be taken as radial. Now one goes through the same steps as before and finds that the temperature and the drift velocity are isotropic. Therefore we shall define $a_1 = a_2 = a_3 = \bar{a}$; $\Theta_1 = \Theta_2 = \Theta_3 = \bar{\Theta}$. Because of the presence of positively charged ions, equation (5) undergoes a change. In order to avoid mix-up with the definitions of the previous section, we shall define

$$\rho(\tau) = \frac{C_e(\tau)}{C_e(0)} \quad \text{and} \quad \chi = \frac{a}{\Omega_e},$$

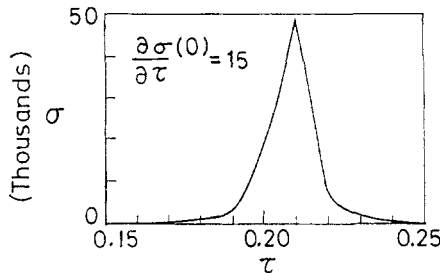


FIG. 5.— τ versus \tilde{A} for $d\sigma/d\tau = 5$.

where Ω_p is the electron plasma frequency corresponding to $C_e(0)$ and τ is in units of Ω_p^{-1} . Hence the new set of equations for the electron dynamics is

$$\frac{\Theta}{\Theta} + 2\chi = 0, \quad (18)$$

$$\dot{\chi} + \chi^2 - \frac{(\rho - \Delta)}{3} = 0, \quad (19)$$

$$\frac{\dot{\rho}}{\rho} + 3\chi = 0, \quad (20)$$

where the dot stands for the derivative with respect to time τ , and Δ is the ion-to-electron density ratio at $\tau = 0$. From (19) and (20), we can eliminate χ to obtain

$$\dot{\rho} - \frac{4}{3} \frac{\dot{\rho}^2}{\rho} + \rho(\rho - \Delta) = 0. \quad (21)$$

It is easy to see that for small deviations about the neutral state of the plasma the electrons oscillate with a plasma frequency corresponding to the density at $\tau = 0$. The above equations cannot be solved on the lines of the previous section. Therefore when either of ρ and $|\dot{\rho}|$ are not small, one would have to resort to numerical methods. However, the first integration of (21) can be accomplished, use of which will be made later on, obtaining

$$\dot{\rho} = [K_0^2 \rho^{8/3} - (6^3 + 3\Delta\rho^2)]^{1/2}, \quad (22)$$

where K_0 is the constant of integration. For subsequent discussion we shall need the expression for the total time derivative of \mathbf{u} , which depends on \mathbf{r} and χ . The expression for \mathbf{r} continues to be given by (15). Thus we have

$$\frac{d\mathbf{u}}{d\tau} = \frac{d}{d\tau}(\chi\mathbf{r}) = -\frac{\omega_0}{3} \frac{d}{d\tau} \left[\frac{\dot{\rho}}{\rho^{4/3}} \right] \mathbf{r}_0, \quad (23)$$

where (15) has been used. We now substitute $\dot{\rho}$ from (22) and obtain

$$\frac{d\mathbf{u}}{d\tau} = \frac{\omega_0}{\rho^{1/3}} (\rho - \Delta) \mathbf{r}_0. \quad (24)$$

Let us consider examples of independently prepared plasma having: (a) different densities with a common value for $\rho(0)$, and (b) different ρ_s with a common value for the ion density. For inter-comparison the above definition of time is suitable for case (a) because $\rho(0)$ is kept constant, while for case (b), since $\rho(0)$ varies, τ will be redefined as $\tau = t\Omega_c$, where Ω_c is the electron plasma frequency corresponding to the common species density. Equations (19)–(20) will be solved for case (a). The result for case (b) can be derived from (a) by simply multiplying the time axis there by

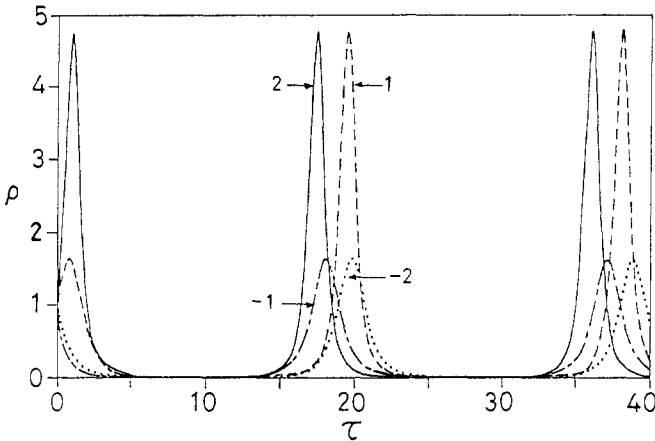


FIG. 6.—For a two-species plasma, τ versus density ρ for various initial values of $d\rho/d\tau$, when $\Delta = 0.1$.

$\Delta = [C(0)/N]^{1/2}$, where N is the ion density. It can be seen by using (18) and (21) that $\Theta = \Theta(0)\rho^{2/3}$, as in equation (17). The above equations for case (a) have been solved for initial values of $\dot{\rho} = \pm 1, \pm 2$, for each case $\Delta = 0.1, 1, 3, 10$, and the results are plotted in Figs 6–13. Since $\mathbf{u} = -\dot{\rho}(\rho^{4/3}/3)\mathbf{r}_0$, we have plotted u_x/x_0 from which \mathbf{u} can be determined. The positive and the negative values of $\dot{\rho}(0)$ respectively correspond to contraction and expansion for the electrons. One observes periodic peaks in the density, but is not simple periodic. However, the drift velocity shows near harmonic periodicity. Further, both ρ and \mathbf{u} exhibit increasing periods as Δ decreases, and vice versa. One also observes large variations in the density as Δ increases, whereas u varies only marginally. Initially, if the electrons are expanding then peaks in the density lag

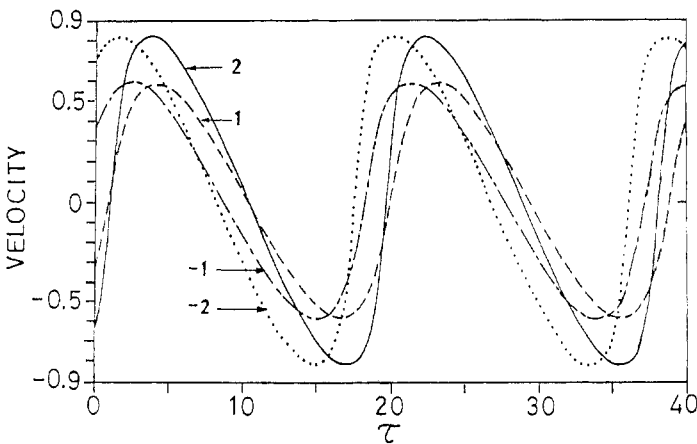
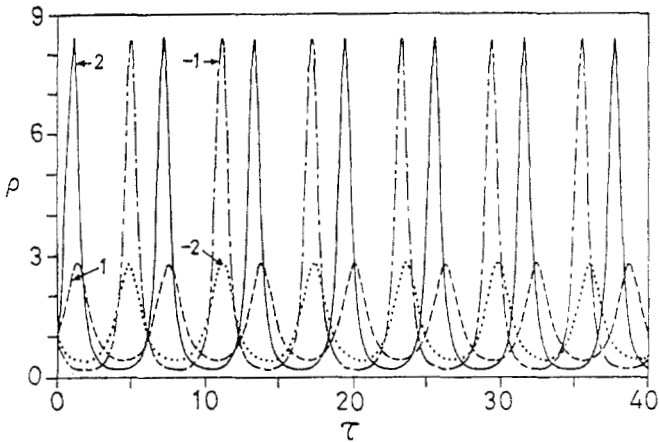


FIG. 7.—For a two-species plasma, τ versus the x -component of the radial velocity normalized by the initial value of the x -coordinate, for $\Delta = 0.1$.

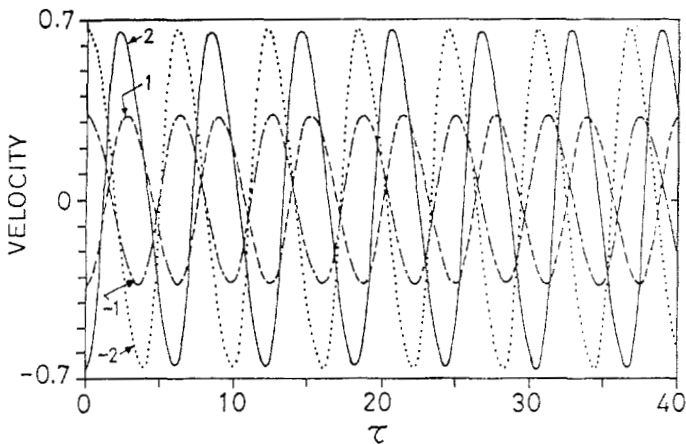
FIG. 8.—Fig. 6, repeated for $\Delta = 1$.

behind the case of initially contracting electrons. The above qualitative behaviour is found to repeat itself for other values of Δ not shown in the diagrams. It is also obvious that the qualitative behaviour in case (b) will be the same as in (a). The relationship of the density-period with Δ can be qualitatively explained. Let us assume, without loss of generality, that $\Delta > 1$.

3.1. *Initially contracting electrons* ($\Delta > 1$)

(a) Whatever is mentioned in this sub-section and subsequently, is going to be based on equation (24) unless otherwise stated. Therefore explicit reference is not going to be made to this equation frequently.

Initially, the force acting on the electrons favours inward motion because of the charge being positive at every point. Therefore ρ continues to increase until at some

FIG. 9.—Fig. 7, repeated for $\Delta = 1$.

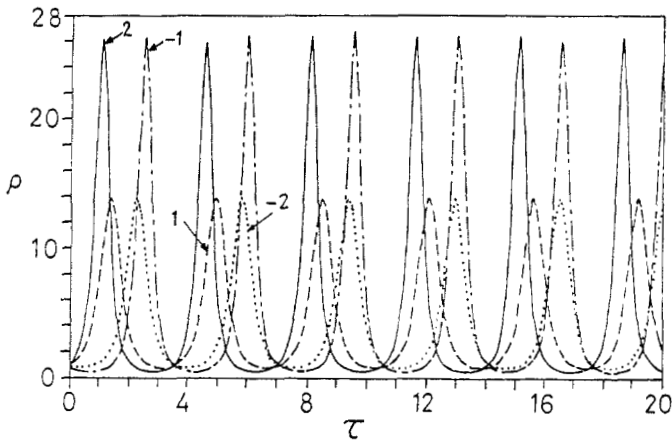


FIG. 10.—Fig. 6, repeated for $\Delta = 3$.

point in time it becomes more than Δ , when the direction of force gets reversed because of the change of sign of the charge density. The inward motion of the electrons is then retarded, resulting in the vanishing of u with ρ attaining its maximum value. The electrons then expand away from the origin, decreasing the density till ρ becomes less than Δ at which point the direction of force is reversed again. The density continues to decrease, reaching its minimum value at $u = 0$. After this, u reverses its sign which causes an inward motion. The electrons then contract towards the origin to create a second peak in the density. The process of recurring maxima and minima in the density continues.

(b) Let us now increase Δ in (a). The initial charge density is now more positive than in (a) and, therefore, the electrons are acted upon by a stronger force towards the origin. The electrons, therefore, will be compressed to a higher density before the

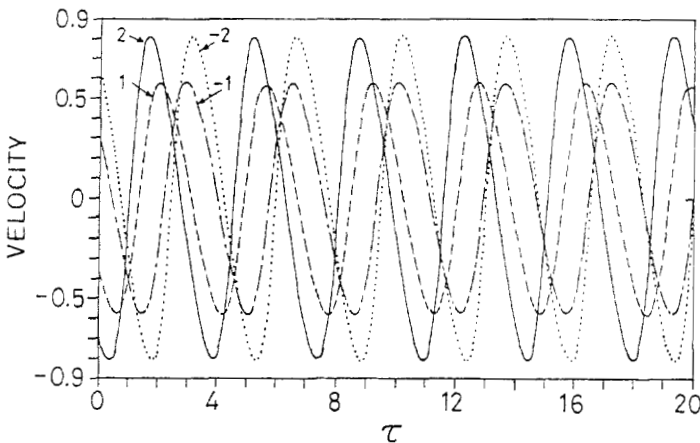
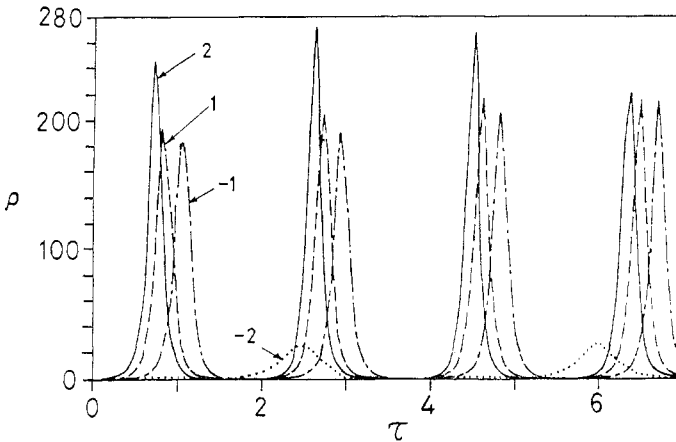
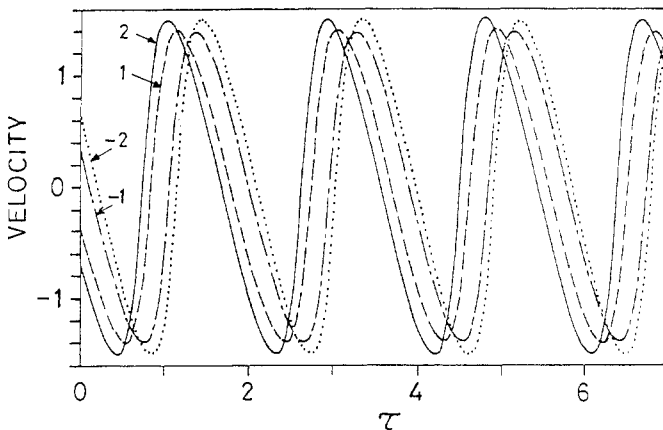


FIG. 11.—Fig. 7, repeated for $\Delta = 3$.

FIG. 12.—Fig. 6, repeated for $\Delta = 10$.

force changes its direction, dragging the electrons to $u = 0$. This will take place in a shorter time than in (a) because $|d\mathbf{u}/d\tau|$ has a higher value at higher densities. The electrons now expand outwards, decreasing the density. When ρ drops below Δ , the force again reverses its direction. Because of the higher value of Δ , the direction of the force is reversed towards the origin at a higher density. As a result the magnitude of the retarding force will also be more, which causes the electrons to spend less time in the expansion phase than in (a) before they move towards the origin to create a second peak in the density. On the lines of the above arguments, it can be easily seen that the second peak will be created in a shorter time than in (a). This process continues. Hence the maximum density will be higher and the period (i.e. the time elapsed in between two consecutive peaks) smaller than in (a), which agrees with the numerical results.

FIG. 13.—Fig. 7, repeated for $\Delta = 10$.

3.2. Initially expanding electrons

Initially the acceleration is against the outward motion which continues till u approaches zero, after which the electrons race towards the origin to form the first peak in the density. In other words, the first peak occurs after the electrons have gone through the expansion–contraction phase as compared with (a), where the first peak occurred during initial contraction phase. Therefore the peak in the present case will be formed later than in (a), which is also born out by the numerical results.

On the lines of the explanation given above it can be seen that if Δ is increased, the time taken by the electrons to move from one peak to another will be less than in (a). Also, at no point in time does Landau damping come into the picture.

4. CONCLUSIONS

Single- and two-species non-neutral plasmas with radial electrostatic field have been studied. Enhancement in the density of the single-species plasmas is observed in the contracting phase. The larger the initial inward radial velocity, the larger the maximum density achieved. The temperature varies as the two-thirds power of the density. After reaching maximum density the plasma expands forever. No Landau damping is observed.

For two-species plasmas where the dynamics of the heavy ions can be neglected, the electrons show large temporal variations in density and temperature when the initial time derivative of the density is increased. The temporal behaviour, which is not simple periodic, shows peaks in the density and the temperature. If the initial ion-to-electron density ratio is increased (decreased), while keeping the initial time derivative of the density fixed, the time elapsed between any two peaks becomes small (large). The mass velocity, which is near periodic in time, does not show as much a strong dependence on Δ as the density and the temperature do.

In this paper we have considered plasmas which are spatially isotropic. If the isotropy is violated, the compression will not be so strong and consequently the peak densities will be lower. There is another question concerning instability. Since the dynamics involved in the contraction and expansion processes is non-linear in nature, weak perturbations will not be able to disturb the equilibria. However, strong perturbation may lead to instabilities.

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