Appendix B

Ion acoustic waves and Langmuir waves

In Section 2.8 the phase velocities of ion and electron acoustic waves were used in a qualitative description of the incoherent scatter spectrum. A simple derivation of the dispersion equations of these two wave modes is given in this appendix.

B.1 Ion acoustic waves

Ion acoustic waves propagating in plasma are nearly similar to ordinary sound waves in neutral gas: they are longitudinal waves consisting of compressions and rarefactions progressing in the medium. The role of ions is the same as that of neutral atoms in ordinary sound waves. A difference is that, unlike sound waves, ion acoustic waves can also propagate in collisionless medium, because the charged ions interact at long distances via their electromagnetic field. A second difference is that plasma also contains electrons which have their effect on the wave dispersion equation. Due to their small mass the electrons are very mobile and they quickly follow the ion motion trying to preserve the charge neutrality. The electron motion is caused by a small electric field internally generated by the plasma as a result of variations in the local ion density.

We assume that the plasma consists of electrons and a single species of positive ions. Their momentum equations read

$$m_{i,e} n_{i,e} \left[\frac{\partial \mathbf{v}_{i,e}}{\partial t} + (\mathbf{v}_{i,e} \cdot \nabla) \mathbf{v}_{i,e} \right] = -\nabla p_{i,e} \pm e n_{i,e} \mathbf{E}. \qquad \text{(B.1)}$$

Here $m_{i,e}$, $n_{i,e}$, $\mathbf{v}_{i,e}$ and $p_{i,e}$ are the mass, number density, velocity and pressure of the ions and the electrons, respectively, e is the positive elementary charge and $\mathbf{E}$ is the electric field. The upper sign is valid in the ion and the lower one in the electron momentum equation. No external electric or magnetic field is assumed; the field $\mathbf{E}$ exists only due to a small charge separation caused by a slightly different motion of the ions and the electrons.

The charge neutrality is preserved if $n_i = n_e = n$, which can only be accomplished when $\mathbf{v}_i = \mathbf{v}_e = \mathbf{v}$. The slight departure from charge neutrality is so small that we can assume these conditions to be valid. Then the momentum equations (B.1) give

$$\frac{1}{m_i}(-\nabla p_i + ne\mathbf{E}) = \frac{1}{m_e}(-\nabla p_e - ne\mathbf{E}), \tag{B.2}$$

which can be solved to obtain the electric field

$$\mathbf{E} = \frac{m_e \nabla p_i - m_i \nabla p_e}{ne(m_i + m_e)} \approx -\frac{1}{ne}\nabla p_e. \tag{B.3}$$

The approximation is valid because the electron mass is much smaller than the ion mass. The ion momentum equation can now be written as

$$m_i n \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p_i - \nabla p_e. \tag{B.4}$$

The ion and electron gases are assumed to obey the ideal gas law and the compressions to be adiabatic. Hence

$$p_{i,e} = n k_B T_{i,e} \tag{B.5}$$

and

$$p_{i,e} = C_{i,e} n^{\gamma}, \tag{B.6}$$

where $T_{i,e}$ are the ion and electron temperatures, k_B is the Boltzmann constant, γ is the ratio of specific heat capacities and $C_{i,e}$ are constants. Eqs. (B.5) and (B.6) readily give

$$\nabla p_{i,e} = \gamma k_B T_{i,e} \nabla n. \tag{B.7}$$

By inserting this in eq. (B.4) we obtain the ion momentum equation in the form

$$m_i n \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\gamma k_B (T_i + T_e)\nabla n. \tag{B.8}$$

In addition to the momentum equation, we also need the ion continuity equation

$$\frac{\partial n}{\partial t} + \nabla\cdot(n\mathbf{v}) = 0. \tag{B.9}$$

Because the ion and electron densities and velocities are identical, this is also the continuity equation of electrons. Here we have assumed that the production and loss of the ionisation cancel, which is a good approximation in the time scale of the wave variations.

Eqs. (B.8) and (B.9) can now be used in deriving the dispersion equation of the ion acoustic wave. We assume no average velocity of the medium so that $\mathbf{v}$ is the velocity associated with the wave motion. Both this velocity and the wave disturbance n' of the ion number density are taken to be small. If the undisturbed ion density is n_0, $n = n_0 + n'$. By inserting this in eqs. (B.8) and (B.9) and linearising the resulting equations we obtain

$$\begin{aligned} m_i n_0 \frac{\partial \mathbf{v}}{\partial t} + \gamma k_B (T_i + T_e) \nabla n' &= 0 \\ \frac{\partial n'}{\partial t} + \nabla \cdot (n_0 \mathbf{v}) &= 0. \end{aligned} \tag{B.10}$$

In the linearisation we have put all second or higher order terms (i.e. terms containing a product of at least two small wave quantities) to zero.

The next step is to consider a plane wave propagating in the medium so that both n' and $\mathbf{v}$ are proportional to $\exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$. Then $\partial/\partial t = i\omega$ and $\nabla = -i\mathbf{k}$. When this is applied to eq. (B.10), the result is

$$\begin{aligned} i\omega m_i n_0 \mathbf{v} \quad - \quad i\gamma k_B (T_i + T_e) n' \mathbf{k} &= 0 \\ -i n_0 \mathbf{k} \cdot \mathbf{v} \quad + \quad i\omega n' &= 0. \end{aligned} \tag{B.11}$$

The former of these equations shows that $\mathbf{v}$ is parallel to the wave vector so that $\mathbf{k} \cdot \mathbf{v} = kv$, which indicates that the wave is longitudinal. Then eq. (B.11) can be put in matrix form

$$\begin{pmatrix} m_i n_0 \omega & -\gamma k_B (T_i + T_e) k \\ -n_0 k & \omega \end{pmatrix} \begin{pmatrix} v \\ n' \end{pmatrix} = 0. \tag{B.12}$$

This equation has non-trivial solutions only if the determinant of the matrix is zero, i.e.

$$\begin{vmatrix} m_i \omega & -\gamma k_B (T_i + T_e) k \\ -k & \omega \end{vmatrix} = 0. \tag{B.13}$$

This gives

$$m_i \omega^2 - \gamma k_B (T_i + T_e) k^2 = 0, \tag{B.14}$$

which leads to the phase velocity of the ion acoustic wave

$$v_+ = \frac{\omega}{k} = \sqrt{\frac{\gamma k_B (T_i + T_e)}{m_i}}. \tag{B.15}$$

The thermal (most probable) speed of electrons is

$$v_{me} = \sqrt{\frac{2 k_B T_e}{m_e}}. \tag{B.16}$$

Obviously, $v_{me} \gg v_+$ so that the electrons can travel many wavelengths during a single oscillation period. Thus the electrons are fast enough to equalise the electron temperature and therefore the electron compressions are isothermal rather than adiabatic. Hence the above theory needs a slight modification. Instead of eq. (B.7), the pressure gradient of the electrons will be

$$\nabla p_e = k_B T_e \nabla n_e, \tag{B.17}$$

which means that the factor $\gamma(T_i + T_e)$ in the above theory should be replaced by $(\gamma T_i + T_e)$. The resulting dispersion equation of ion acoustic waves is

$$v_+ = \frac{\omega}{k} = \sqrt{\frac{k_B(\gamma T_i + T_e)}{m_i}}. \tag{B.18}$$

Still another point to consider is the value of γ associated with the compressions of the ion gas. A more advanced treatment would actually be needed in determining its value; here we only take the following heuristic approach. In a three-dimensional case the gas would have three degrees of freedom, and then the proper value would be $\gamma = 5/3$. When the collision frequency is small, however, the gas may not have enough time to distribute the changes of thermal energy to all degrees of freedom, and then the situation is effectively one-dimensional. This means that the number of degrees of freedom is reduced to one and, consequently, $\gamma = 3$.

B.2 Langmuir waves

One might think that a thermally excited wave mode similar to the ion acoustic wave, with its phase speed determined by the electron mass and temperature, could propagate in the electron gas of the plasma. In thermal equilibrium, however, the electron motion is strongly controlled by ions, and therefore no such wave with a significant amplitude is generated. The situation will change if energetic photoelectrons or energetic electrons caused by particle precipitation are present in the plasma. Then the electrons may have a sufficient energy to create density enhancements violating the local charge neutrality and electron plasma waves with associated electric fields may be generated. The frequency of these waves is so high that the massive ions are not affected by them and therefore the ion gas can only be regarded as a constant positive charge density taking care of average charge neutrality in the medium. One should notice that, due to the significant electric field, the wave mode thus generated has characteristics different from those of the ion acoustic wave.

With the help of eq. (B.7), the electron momentum equation in eq. (B.1) can be written as

$$m_e n_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla)\mathbf{v}_e\right] = -\gamma k_B T_e \nabla n_e - e n_e \mathbf{E}. \tag{B.19}$$

In addition, the electron density must obey the continuity equation

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0 \tag{B.20}$$

and the electric field the Gauss law

$$\nabla \cdot \mathbf{E} = e(n_i - n_e)/\varepsilon_0. \tag{B.21}$$

If the ion density is not affected by the rapidly oscillating wave, obviously $n_e - n_i = n'_e$ is the wave disturbance in the electron density. Then $n_e = n_{0e} + n'_e$, where $n_{0e} = n_i$ is the average electron density. When $\mathbf{v}_e$ and $\mathbf{E}$ are also small wave quantities, the linearisation of eqs. (B.19)–(B.21) gives

$$\begin{aligned} m_e n_{0e}\frac{\partial \mathbf{v}_e}{\partial t} + \gamma k_B T_e \nabla n'_e + e n_{0e}\mathbf{E} &= 0 \\ \frac{\partial n'_e}{\partial t} + n_{0e}\nabla\cdot\mathbf{v}_e &= 0 \\ \nabla\cdot\mathbf{E} + \frac{e}{\varepsilon_0}n'_e &= 0. \end{aligned} \tag{B.22}$$

Taking all wave quantities to be proportional to $\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]$, we can further write

$$\begin{aligned} i m_e n_{0e}\omega\mathbf{v}_e - i\gamma k_B T_e n'_e\mathbf{k} + e n_{0e}\mathbf{E} &= 0 \\ -i n_{0e}\mathbf{k}\cdot\mathbf{v}_e + \quad i\omega n'_e \quad &= 0 \\ \frac{e}{\varepsilon_0}n'_e - i\mathbf{k}\cdot\mathbf{E} &= 0. \end{aligned} \tag{B.23}$$

The latter two of these equations deal with the components of $\mathbf{v}_e$ and $\mathbf{E}$ parallel to $\mathbf{k}$, and the perpendicular component of the first equation only gives the velocity of an electron in an oscillating electric field. Therefore only the parallel component of the first equation is of interest and we can take both $\mathbf{v}_e$ and $\mathbf{E}$ to be parallel to $\mathbf{k}$. Then $\mathbf{k}\cdot\mathbf{v}_e = kv_e$ and $\mathbf{k}\cdot\mathbf{E} = kE$, and eqs. (B.23) can be written in matrix form

$$\begin{pmatrix} i m_e n_{0e}\omega & -i\gamma k_B T_e k & e n_{0e} \\ -n_{0e}k & \omega & 0 \\ 0 & e/\varepsilon_0 & -ik \end{pmatrix}\begin{pmatrix} v_e \\ n'_e \\ E \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{B.24}$$

The dispersion equation is again obtained by putting the determinant of the matrix to zero, i.e.

$$\begin{vmatrix} i m_e\omega & -i\gamma k_B T_e k & e n_{0e} \\ -k & \omega & 0 \\ 0 & e/\varepsilon_0 & -ik \end{vmatrix} = 0, \tag{B.25}$$

which gives

$$\omega^2 = \frac{n_{0e}e^2}{\varepsilon_0 m_e} + \frac{\gamma k_B T_e}{m_e}k^2. \tag{B.26}$$

In terms of the angular plasma frequency ω_p and the electron thermal speed v_{me} this is equivalent to

$$\omega^2 = \omega_p^2 + \frac{\gamma}{2}v_{me}^2 k^2. \tag{B.27}$$

The waves obeying this dispersion equation are called electron plasma waves, Langmuir waves or electron acoustic waves. Their frequency is always higher than the plasma frequency. If $\omega_p^2 \ll \frac{\gamma}{2}v_{me}^2 k^2$, the phase velocity is

$$v_- = \frac{\omega}{k} = \sqrt{\frac{\gamma k_B T_e}{m_e}}, \tag{B.28}$$

which is of the same form as the phase velocity of the ordinary sound wave in neutral gas. Therefore the term electron acoustic wave is sometimes reserved for this high-frequency approximation.

Unlike in ion acoustic waves, the electron gas is in this case adiabatic rather than isothermal. This is because the frequencies of the waves are much higher. When the electron collision frequency is small, one can show that the electrons gas behaves as if it would have only a single degree of freedom. Then $\gamma = 3$, and eq. (B.27) can be written as

$$\omega^2 = \omega_p^2 + \frac{3}{2} v_{me}^2 k^2. \tag{B.29}$$

This is known as the Bohm-Gross dispersion relation. Because $v_{me}^2/\omega_p^2 = 2\lambda_D^2$, the phase velocity is

$$v_- = \frac{\omega}{k} = \omega_p \sqrt{1/k^2 + 3\lambda_D^2}, \tag{B.30}$$

which is equivalent to eq. (2.99). Here λ_D is the Debye length.

In the presence of energetic electrons the electron acoustic waves are excited and the plasma lines appear on both sides of the ion line in the incoherent scatter spectrum, as explained in Section 2.8. At wavelengths much longer than the Debye length the frequency of the electron acoustic waves is equal to the plasma frequency. This gives a possibility to determine the electron density from the frequency of the plasma line.